Hamiltonian Results for Spacecraft with Electric Propulsion System and Trajectory Optimization

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Abstract

In this paper we focus on Hamiltonian results to solve the motion equations for spacecraft with electric propulsion system. Once a path has been chosen, we perform the next step: path solving. Employing a shape-based approach we perform a broad search over the design space to discover suboptimal Low Thrust Gravity-Assist (LTGA) trajectories.

Keywords: Trajectory optimization, Spacecraft equations of motion, Low Thrust Gravity-Assist, Rocket equation.

Introduction

Trajectories to Pluto with reasonably short flight times, however, generally require high flyby velocities at Jupiter often resulting in heliocentric escape [3]. Although Ref. [3] did not address LTGA trajectories, the SEP engine’s inability to thrust as far out as Jupiter means that a high $V_\infty$ at Jupiter is required in our case, as well.

Motion equations for spacecraft with electric propulsion system can be written as

$$\ddot{x} = -\frac{\mu_G}{r^3} + \frac{T}{m} \cos \alpha \cos \alpha_{\text{xy}}$$

$$\ddot{y} = -\frac{\mu_G}{r^3} + \frac{T}{m} \cos \alpha \sin \alpha_{\text{xy}}$$

$$\ddot{z} = -\frac{\mu_G}{r^3} + \frac{T}{m} \sin \alpha$$

$$\dot{m} = -c$$

The equations of motion can be written in first-order, non-dimensional system as

$$\dot{x} = v_x$$

$$\dot{y} = v_y$$

$$\dot{z} = v_z$$
\[\dot{v}_x = -\frac{x}{r^3} + \tau \cos \alpha_z \cos \alpha_{xy}\] (8)

\[\dot{v}_y = -\frac{y}{r^3} + \tau \cos \alpha_z \sin \alpha_{xy}\] (9)

\[\dot{v}_z = -\frac{z}{r^3} + \tau \sin \alpha_z\] (10)

And the corresponding Hamiltonian results

\[
H = \lambda_1 v_x + \lambda_2 v_y + \lambda_3 \Gamma_z + \lambda_4 \left(-\frac{x}{r^3} + \tau \cos \alpha_z \cos \alpha_{xy}\right) + \\
\lambda_5 \left(-\frac{y}{r^3} + \tau \cos \alpha_z \sin \alpha_{xy}\right) + \lambda_6 \left(-\frac{z}{r^3} + \tau \sin \alpha_z\right)
\] (11)

The corresponding costate equations are then obtained as:

\[
\dot{\lambda}_1 = \lambda_4 (y^2 + z^2 - 2x^2) / r^5 - 3\lambda_3 y + \lambda_6 z / r^5
\] (12)

\[
\dot{\lambda}_2 = \lambda_5 (x^2 + z^2 - 2y^2) / r^5 - 3\lambda_4 x + \lambda_6 z / r^5
\] (13)

\[
\dot{\lambda}_3 = \lambda_6 (y^2 + x^2 - 2z^2) / r^5 - 3\lambda_4 y + \lambda_5 y / r^5
\] (14)

\[
\dot{\lambda}_4 = -\lambda_1
\] (15)

\[
\dot{\lambda}_5 = -\lambda_2
\] (16)

\[
\dot{\lambda}_6 = -\lambda_3
\] (17)

\[
\dot{\lambda}_7 = -\lambda_4 \cos \alpha_z \cos \alpha_{xy} - \lambda_5 \cos \alpha_z \sin \alpha_{xy} - \lambda_6 \sin \alpha_z - 2\tau \lambda_7 k
\] (18)

The first derivative of the Hamiltonian with respect to \(\alpha_z\) results
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\[
\frac{\partial H}{\partial \alpha_z} = 0 = -\lambda_4 \tau \sin \alpha_z \cos \alpha_{xy} - \lambda_5 \tau \sin \alpha_z \sin \alpha_{xy} + \lambda_6 \tau \cos \alpha_z
\]  
(19)

which simplifies to

\[
\sin \alpha_z = \frac{\lambda_6}{\lambda_4 \cos \alpha_{xy} + \lambda_5 \sin \alpha_{xy}}, \quad \text{with } \alpha_z \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]
\]  
(20)

Accordingly, the second-order derivative of the Hamiltonian with respect to \( \alpha_z \) can be written as

\[
\frac{\partial^2 H}{\partial \alpha_z^2} = -\lambda_4 \cos \alpha_z \cos \alpha_{xy} - \lambda_5 \tau \cos \alpha_z \sin \alpha_{xy} - \lambda_6 \tau \sin \alpha_z > 0
\]  
(21)

or equivalently

\[
\left(\lambda_4 \cos \alpha_{xy} + \lambda_5 \sin \alpha_{xy}\right) \cos \alpha_z + \lambda_6 \sin \alpha_z < 0
\]  
(22)

Substituting for the costate using equations yields to independent constraints in the form

\[
\frac{\lambda_6 \cos^2 \alpha_z}{\sin \alpha_z} + \lambda_5 \sin \alpha_z = \frac{\lambda_6}{\sin \alpha_z} < 0 \rightarrow \text{sign} \left(\frac{\lambda_6}{\sin \alpha_z}\right) = -\text{sign} \left(\sin \alpha_z\right)
\]  
(23)

and similarly,

\[
\left(\lambda_4 \cos \alpha_{xy} + \lambda_5 \sin \alpha_{xy}\right) \cos \alpha_z < 0 \rightarrow \text{sign} \left(\lambda_4 \cos \alpha_{xy} + \lambda_5 \sin \alpha_{xy}\right) = -\text{sign} \left(\cos \alpha_z\right)
\]  
(24)

The corresponding control law for \( \alpha_z \) then results

\[
\dot{\alpha}_z = \tan 2 \left(\frac{-\lambda_6}{\left(\lambda_4 \cos \alpha_z + \lambda_5 \sin \alpha_{xy}\right)}\right), \quad \text{with } \alpha_z \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]
\]  
(25)

For the control angle \( \alpha_{xy} \) the first order derivative of the Hamiltonian results
The corresponding second order derivative is obtained as

$$\frac{\partial H}{\partial \alpha_{xy}} = -\lambda_4 \tau \cos \alpha_z \cos \alpha_{xy} - \lambda_4 \tau \sin \alpha_z \sin \alpha_{xy} > 0$$

(27)

As we noted earlier $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \rightarrow \cos \alpha > 0$. Using equation (26) two independent constraints for the control angle $\alpha_{xy}$ can be obtained as

$$\text{sign}(\lambda_4) = -\text{sign}(\sin \alpha_{xy}) \quad \text{and} \quad \text{sign}(\lambda_4) = -\text{sign}(\cos \alpha_{xy})$$

(28)

Therefore the correct control law for $\alpha_{xy}$ is obtained

$$\alpha_{xy}^* = a \tan 2 \left( \frac{-\lambda_4}{\lambda_4} \right), \text{ with } \alpha_{xy}^* \in [0, 2\pi)$$

(29)

Note that mixed second order derivatives of the Hamiltonian add no additional information.

**Trajectory Optimization**

The equation of an arc in applied thrust and polar form, can be given as

$$r = k_0 \exp[k_1 \sin(k_2 \theta + \phi)]$$

(30)

Where $k_0$, $k_1$, $k_2$ and $\phi$ are constants. To evaluate trajectories found by STOUR, we employ a cost function that computes the total propellant mass fraction due to the launch energy, thrust-arc propellant, and arrival $V_{\infty}$ (if a rendezvous is desired). Tsiaolkovsky's rocket equation is used to account for the departure and arrival energies [8, 9]. For the launch $V_{\infty}$ ($\Delta V_1$ in Eq. 31), we use a specific impulse of 350 seconds ($I_{sp1}$) to represent a chemical launch vehicle. A low-thrust specific impulse ($I_{sp2}$) of 3000 seconds is applied to the arrival $V_{\infty}$ ($\Delta V_2$) because we assume that the low-thrust arcs will remove any excess velocity in rendezvous missions. Finally, the thrust-arc propellant mass fraction given by STOUR ($p_{mf}$) yields the total
propellant mass fraction \( t_{mf} \):

\[
t_{mf} = 1 - (1 - p_{mf}) \exp \left( \frac{-\Delta V_1}{gI_{sp1}} \right) \exp \left( \frac{-\Delta V_2}{gI_{sp2}} \right)
\]  

(31)

For flyby missions, \( \Delta V_2 = 0 \). While Eq. 31 is only an approximation of the true propellant costs, it serves quite adequately to reduce the candidate trajectories to a manageable number among the myriad (up to tens of thousands) of possibilities produced by STOUR [10].

Conclusion

We have discussed the two-dimensional trade studies are in good agreement with results obtained for the three-dimensional analysis. We optimized our trajectories to maximize the final spacecraft mass, which in turn may increase the scientific value of a mission.

References


